ABOUT IRR AND NEW INVESTMENT PRIORITY INDEX

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Abstract. The two most important criteria are the net present value (NPV) and the internal rate of return (IRR) for choosing among investment projects. The analysis of IRR and NPV indicates an unequivocal choice among the criteria NPV and IRR. We prepare the IRR rule. The static investment optimization problem of IRR is solved as the optimal usefulness of the well-known mathematical result of the knapsack problem. The word "static" means that all projects under consideration have equal duration. The dynamic optimization problem of investments is solved in the case of one-time costs by obtaining results at different points in time (distributed lags), namely, by the fragmentation of the project. The key result is the new investment priority index offering a simple formula for optimal choice among "one-time investments and distributed lags" projects by splitting these projects by time. The procedure how to split the project by time is given. The most general case - distributed in time costs and results - can be built with approximate heuristic solutions. For discussion purposes, a relatively simple investment project is considered numerically.

Keywords: investment, net present value, internal rate of return, investment priority index.

Introduction

In [1], we compared the two most well-known criteria for investment performance measurement: IRR (Internal Rate of Return) – internal rate of return and NPV (Net Present Value) – discounted total profit and showed preference for the IRR criterion over the criterion NPV. In many circumstances, both criteria rank investment projects in the same order. In some situations, however, the two criteria provide different rankings. The statement "IRR performs better than NPV", honestly speaking, is an unsolved question really and is relevant to this day. This inconsistency sparked a debate about which criterion is better. The debate lasted over 100 years (see [2-4]). Proof of this is the abundance of scientific papers: Google Scholar gives even 28400 references on request for "IRR vs NPV mathematical analysis".

Let us give a mathematical definition of NPV and IRR. If Q_t is the expected return on investment, i.e. the difference between revenue and costs in the t-th year, T is the settlement period, and E is the amount of the discount interest, then the discounted total profit NPV is determined by the formula:

$$NPV = \sum_{t=0}^{T} Q_t \frac{1}{(1+E)^t} . \tag{1}$$

For the same investment project, its rate of efficiency (internal rate of return IRR) is defined as the root of the equation

$$\sum_{t=0}^{T} Q_t \frac{1}{(1+IRR)^t} = 0 . \tag{2}$$

K. K. Seo [5] wrote that approximately 55% of business persons evaluate investment projects with the help of IRR and only 10% − with the help of NPV. The detailed discussions about IRR can be found in [6-9]. In this sense, noteworthy is the work of Kannapiran Arjunan [10]. This article presents current evidence for determining the appropriate investment criterion (IRR vs. NPV), focusing on the controversial reinvestment assumption, multiple, negative, zero or no IRR, mutually exclusive investments, as well as independent projects. These results consistently support the following statement: IRR is the best criterion for accepting, rejecting, or ranking mutually exclusive projects as well as independent projects. Net present value will continue to be useful in other areas for estimating the present value. The proposed new method solved most of the problems such as reinvestment, better ranking of mutually exclusive projects by IRR, and the need to revise NPV rules. At the same time, this shows that a clear solution has not yet been found. In the following, we discuss the IRR rule.

Materials and methods

Static investment optimization problem. Let us start with the famous mathematical problem about a knapsack and show that its economic interpretation gives solving a static investment optimization problem.

Let there be a knapsack of the volume Z, and also there are N items: the volume of each item Z_i and its usefulness P_i, $i = 1, 2, \ldots, N$. It is required to select the optimal composition of a knapsack from this set items such that

$$\sum_{i \epsilon l} P_i = max \{l \epsilon (1, 2, \ldots, N)\} \quad (3)$$

given that $\sum_{i \epsilon I} Z_i \leq Z$.

If necessary, crushing of the item is allowed, you can take a part of an object with a volume αZ_i where $0 < \alpha < 1$, considering that its utility is equal to αP_i.

Theorem I. The knapsack problem. The optimal usefulness of the knapsack is provided by the following procedure:

a) items should be arranged in descending order of their specific utility P_i/Z_i;

b) considering that this order coincides with the growth index i and if for the first l items the following holds: $\sum_{i=1}^{l} Z_i = Z$ condition, then the required set is $\{1, 2, \ldots, l\}$;

c) if $\sum_{i=1}^{l-1} Z_i < Z < \sum_{i=1}^{l} Z_i$, then the optimal set consists of the first $l - 1$ items, to which part of an item l is added measured by αZ_l, where $\alpha = (Z - \sum_{i=1}^{l-1} Z_i)/Z$.

The proof of the theorem follows from arguments from the contrary.

Economic interpretation. If by Z we mean available capital (maximum possible costs), under subjects to understand investment projects, which require expenses Z_i and are capable of producing results of size P_i at the same time interval, for example, after a year, $i = 1, 2, \ldots, N$, then the optimization problem of knapsack is equivalent to choosing such a maximum number l of investment projects from the list of projects, ordered in descending order of relationship P_i/Z_i, so that total costs do not exceed the cash capital, i.e. $\sum_{i=1}^{l} Z_i \leq Z$.

Thus, the simplest problem of investment optimization, which we call a static task, is mathematically strictly proven. The word "static" is used here in the sense of one-time, since all projects under consideration have equal duration (say, a year; although it may be another period, it is only important that it was the same for all projects).

Example. To illustrate Theorem I, we present a numerical example. Let there be one unit of money (for example, a million euros), three projects are given, differing in costs and results: $Z_1 = 1$; $P_1 = 3$; $Z_2 = 0.6$; $P_2 = 2.5$; $Z_3 = 0.4$; $P_3 = 2$ (see Fig. 1). Results are obtained after the billing period $T = 1$ year.

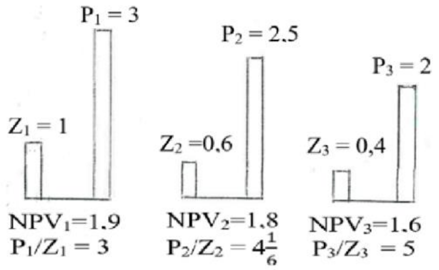

Fig. 1. **Comparison of three investment options at $Z = 1$**

According to formula (1), projects should be ordered into order 1, 2, and 3, since $NPV_1 = 1.9$; $NPV_2 = 1.8$; $NPV_3 = 1.6$, and the first project is subject to implementation. The solution to the knapsack problem (Theorem 1) gives the reverse order of projects $P_1/Z_1 < P_2/Z_2 < P_3/Z_3$, since $P_1/Z_1 = 3$; $P_2/Z_2 = 4$; $P_3/Z_3 = 5$, i.e. the most profitable is the third project, and the available unit of money should be invested in the third and second projects. This solution allows at the same costs $Z_2 + Z_3 = 1$ to get results $P_2 + P_3 = 4.5$, not $P_1 = 3$ as in the first case.

Comment. The solution found can be seen directly from the analysis of the source data, since in it it is easy for the mind to go through the possible options. In general, it is not possible to select options. Regular optimal selection procedure investment is required, which is what Theorem 1 provides.

Dynamic optimization problems of investments. *Setting goals*. The original dynamic task of investment optimization will be understood as follows way:

a) there are several investment projects with given economic descriptions of costs and results over time;

b) costs and results are expressed in monetary term units;

c) there is a known initial capital at the moment $T = 0$;

d) time is discrete; the unit of time is a year;

e) investment projects are mutually independent;

g) for each $T > 0$ is known as the sum of current hidden (for unfinished projects) and explicit results;

h) it is necessary to select such projects so that they minimize the time to reach a given state (the sum of the current hidden and explicit results) or maximize the specified amount for each current moment T.

Investment projects can be divided into three groups in order of increasing complexity of their description:

a) one-time costs and receipt of one-time results after a certain time interval, which is called lag; we will call such projects simple;

b) one-time costs with obtaining results at different points in time (distributed lags);

c) most general case – distributed in time, costs and results.

Investment priority index. Let us introduce the basic concepts. Let a simple project be given: one-time costs Z and one-time results P obtained after τ years. Let us denote such a project by a triple of numbers (Z, τ, P), and let us introduce the index γ, proposed for the first time and determined by the formula

$$\gamma = (P/Z)^{1/\tau} \tag{4}$$

and use it further to characterize the priority of this simple project among other simple projects.

Let us give a mathematical justification for this index. According to formula (2), with the notation $\gamma = 1 + IRR$ we have the equation $-Z + P\,(1/\gamma^{\tau}) = 0$, it follows after algebraic transformations (4). Thus, for any project of type "one-time costs and one-time results obtained after fixed years" we have the key formula

$$IRR = (P/Z)^{1/\tau} - 1 \tag{5}$$

Below to calculate the hidden results you will need to consider unfinished projects, for example, the project (Z, τ, P) at some moment $\tau_1 < \tau$.

Theorem 2. *On the fragmentation of the project*. For each project (Z, τ, P) at the moment $\tau_1 < \tau$ we can compare two projects (Z, τ_1, P_1) and (Z_2, $\tau - \tau_1$, P), where

$$P_1 = Z_2 = Z\,(P/Z)^{\tau_1/\tau} \tag{6}$$

and we will call such an operation splitting the project into time.

The proof of statement (6) consists of two steps.

First step. Let us give the rationale for this operation, it is illustrated in Fig. 2. For any τ_1 from the region $0 < \tau_1 < \tau$ and given Z from (5) we have

$$P_1 = Z\gamma^{\tau_1} = Z\,(P/Z)^{\tau_1/\tau} \tag{7}$$

which proves the second half of (6).

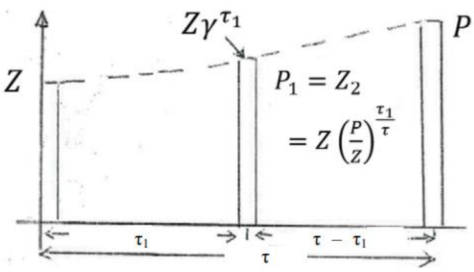

Fig. 2. **Splitting the project by time**

Second step. The value of P_1 will be understood as the hidden results of an unfinished project (Z, τ_1, P_1), in other words, we will assume that in the security market papers, the results of the project (Z, τ, P) at the time τ_1 have the cost P_1. For the remaining part, that is, for the project (Z_2, $\tau - \tau_1$, P) the hidden results of P_1 will be considered as the initial costs Z_2, which proves the first half of the statement (6).

Basic dynamic problem. Given N simple projects (Z_i, τ_i, P_i), which are characterized by costs Z_i, results in P_i and lags τ_i, $i = 1, 2, \ldots, N$, Dan's initial capital Z. The results obtained can be used to implement any of the remaining projects. When choosing an investment decision, it is allowed to split projects according to costs and time.

It is necessary to choose such a strategy for selecting projects at each subsequent moment of obtaining results, with which at each moment T the sum of hidden and explicit results (considering the possibility of splitting according to time) would be the maximum.

Theorem 3. To obtain an optimal solution of the basic dynamic task:

a) it is enough to order these projects (Z_i, τ_i, P_i), $i = 1, 2, \ldots, N$, in descending order of exponents

$$\gamma_i = (Pi/Zi)^{1/\tau i} \quad (8)$$

b) at every current moment in time (including initial) consider the optimal set of projects from the ordered list by item (a) considering the possibility of splitting according to costs the last of the selected projects;

c) the amount of costs at each moment obtaining results should be equal to the sum of these results considering the time division of unfinished projects.

The proof of the theorem should be carried out by contradiction.

One-time costs – multiple results. This is a task with distributed lags. It is possible to reduce it to the basic dynamic problem by crushing projects at costs in the proportion of determined results brought to the initial moment, and namely, let us be given a complex project with disposable costs Z and reusable results P_i through lags τ_1, $i = 1, 2, \ldots, n$. Let us write this in the form (Z; τ_1, P_1; …; τ_n, P_n). We reduce such a complex project to n simple projects: (Z_i, τ_i, P_i), where these are all n simple projects that have the same value γ, determined from

$$Z - \sum_{i=1}^{n} P_i \gamma^{-\tau i} = 0 \quad (9)$$

Figure 3 illustrates the described procedure fragmentation of the project "one-time costs – multiple results".

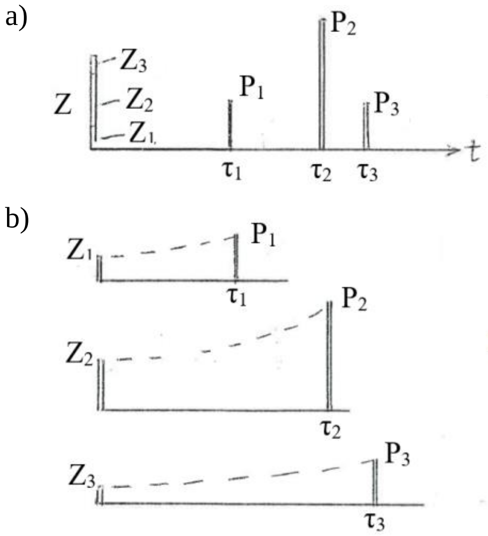

Fig. 3. **Fragmentation of the project with one-time costs and distributed lags:**
a – initial complex project; b – three new simple projects

Multi-time investments. This is the third type of investment projects. It is not directly reducible to the basic dynamic problem and for it the optimal procedure should be sought by solving the optimization problem as a whole for a given investment project. Based on solving the basic dynamic problem in the case of reusable (distributed) investments it can be built with approximate heuristic solutions. In any case, the multi-time investment problem is for future work.

Results and discussion

Let us consider a relatively simple investment project (Table 1). The composition of the project is shown in the following section Project Conditions.

Table 1

Project Conditions

Optimal production volume	100 000 units
Selling price	34 EUR per unit
Cost of materials	16 EUR per unit
Salary costs	3 EUR per unit
Advertising costs	4 EUR per unit
Management costs	4 EUR per unit
Investment period	10 years
Loan amount	3 000 000 euros
Interest rate for credit	7%
Investor investment	1 000 000 euros

The preliminary analysis shows that the brutto income per unit is equal to 7 euros. It follows from the following calculation: the selling price is equal to 34 euros but the production costs (materials + salary + advertising + management) are equal to 27 euros. Note the net sales for the first year are 60% of the optimal production volume.

Table 2

Cash surplus for the project (in thousand EUR)

Years	1	2	3	4	5	6	7	8	9	10
Net sales	2040	3400	3400	3400	3400	3400	3400	3400	3400	3400
Cost of materials	960	1600	1600	1600	1600	1600	1600	1600	1600	1600
Labor costs	180	300	300	300	300	300	300	300	300	300
Advertising costs	240	400	400	400	400	400	400	400	400	400
Loan repayment	300	300	300	300	300	300	300	300	300	300
Management costs	400	400	400	400	400	400	400	400	400	400
Interest payments	210	189	168	147	126	105	84	63	42	21
Marketing costs	700	-	-	-	-	-	-	-	-	-
Cash balance at the beginning of the period	0	50	261	493	746	1020	1315	1631	1968	2326
Cash balance at the end of the period (Q_t)	50	261	493	746	1020	1315	1631	1968	2326	2705
Cash flow	50	211	232	253	274	295	316	337	358	379

How to calculate the IRR value? This is well known. Based on Table 2, we refer to the formula (2) and insert numerical values of the cash balance values Q_t, thus we get IRR = 18.82%.

For discussion purposes, we omit the analysis of Table 2, as it is well-known from tutorials. Our main goal of the paper is to show the use of the firstly proposed investment priority index (formula (4)). To illustrate this, we admit that the investor has no income from cash balance (Q_t) per year. In case of non-receipt of profit from these annual incomes, the investor receives after 10 years 2.705 million euros only from his contribution of 1 million euros. Therefore, according to formula (5)

$$IRR = (2705/1000)^{0.1} - 1 = 10.46\%.$$

The essential here is the uncertain use of annual incomes. Let us note formulas (2) and (5). To get IRR value from (2) is quite sophisticated. It is possible to simplify the calculation process of the case study if one splits the considered investment project into ten simple projects according to the procedure shown in Figure 2.

Conclusions

The static investment optimization problem is solved as the optimal usefulness of the knapsack. The word "static" means that all projects under consideration have equal duration.

The dynamic optimization problem of investments is solved in a particular case – of one-time costs by obtaining results at different points in time (distributed lags), namely, by the fragmentation of the project based on the original investment index (formula 4). This is the key result of the paper. The investment priority index offers a simple formula for optimal choice among “one-time investments and distributed lags” projects. The solution is given by splitting these projects by time. This procedure is given as Theorem 2 “On fragmentation of the project”.

The most general case - distributed in time costs and results - can be built with approximate heuristic solutions (not yet solved in a simple form).

A relatively simple investment project is considered numerically. It shows one more paradox of IRR use for investment project ranging. The essential here is the uncertain use of annual incomes.

Author contributions

Conceptualization, M.SS.; methodology, M.SS, and D.S.; software, D.S.; writing – review and editing, M.SS, and D.S. All authors have read and agreed to the published version of the manuscript.

References

[1] Sneps-Sneppe M. Use of criteria NPV and IRR for choosing among investment projects. 22nd International Scientific Conference Engineering for Rural Development Proceedings. 24 May 2023, pp. 745-750.

[2] Kharabe P., and Rimbach A. MRR, IRR, and NPV as project tranking measures. Real Estate Review, 19(2), 1989, 74.

[3] McDaniel, et at. Discounted cash flow with explicit reinvestment rates: Tutorial and extension. The Financial Review, 23(3), 1988, pp. 369-385.

[4] Beaves R. G. Net present value and rate of return. Implicit and explicit reinvestment assumptions. The Engineering Economist, 33(4), 1988, pp. 275-302.

[5] Seo K.K. Managerial Economics. Irwin Professional Publishing, U.S., 1992.

[6] Zhang Xiaoli. The Choice between NPV and IRR in Investment Decisions. Journal of Central University of Finance & Economics, 8, 2006, pp. 88-92

[7] Magni C.A. The Internal Rate of Return Approach and the AIIKR Paradigm: A Reputation and a Corroboration. The Engineering Economist, 58, 2013, pp. 73-111

[8] Srinivasa Mugeraya. Solutions to the deficiencies of IRR and NPV, including multiple IRRs. SSRN Electronic Journal, 2023. DOI:10.2139/ssrn.4483819

[9] Lin J. A comparative study on the application of NPV and IRR in financial market investment decisions. Academic Journal of Business & Management, 5(4), 2023, pp. 51-54.

[10] Arjunan K. A New Method to Estimate NPV and IRR from the Capital Amortization Schedule and the Advantages of the New Method, Australasian Accounting, Business and Finance Journal, 16(6), 2022, pp. 23-44.